Applied statistics in mineral exploration

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Abstract

Applied statistics provides scores of powerful tests and techniques that are widely used and abused in science and engineering. The variance is the basic measure for variability, precision and risk. The formulas for the variances of the arithmetic mean and the weighted average are derived from the variance of a general function, as defined in probability theory. The requirement of functional or mathematical independence in probability theory translates into the concept of degrees of freedom in applied statistics. Geostatistics is the only variant of statistics in which functional dependence and degrees of freedom are irrelevant. This paper examines whether such basic elements of probability and statistics can be ignored with impunity.

Introduction

The variance is the fundamental measure for random variations in stochastic variables and stochastic systems. Variances are amenable to mathematical analysis. The properties of variances are derived from the variance of a general function. The arithmetic mean and the weighted average are both functionally dependent variables of sets of measured values. The central limit theorem defines the relationship between the variance of a set of measured values and the variance of the arithmetic mean of the set.

The variance of the weighted average, despite its significance in mining, metallurgy and exploration, is not as well documented as the central limit theorem. As a result, kriging variances and covariances of sets of kriged estimates (distance weighted averages) could evolve into the essence of geostatistics.

Dependence and independence

Functional or mathematical dependence. The arithmetic mean is a function of a set of measured values with identical weight factors, and the weighted average is a function of a set of measured values with different weight factors. The requirement of mathematical or functional independence invalidates the variance and covariance of a set of distance weighted averages. The concept of degrees of freedom in applied statistics is the corollary of the requirement of functional independence in probability theory. Degrees of freedom are awarded when values are measured but not when they are calculated from measured values. It is beyond the scope of this paper to prove that adding functionally dependent values to a set of measured values does not add degrees of freedom to the set.

Because degrees of freedom are fundamental in applied statistics, they occur not only in the formulas for variances but also in many statistical tables. For example, the denominators in the formulas for the variances of randomly distributed and ordered sets of measured values are degrees of freedom (Gy, 1979; Merks 1993). Degrees of freedom also occur in the basic formula for the covariance, but they cancel in a simplified formula. However, covariances cannot be tested for statistical significance without taking degrees of freedom into account.

Associative dependence. Associative dependence exists if the degree of association between two or more variables is statistically significant. An example in mineral exploration is associative dependence between grades and densities in massive sulfides. A significant degree of associative dependence permits regression parameters to be used to calculate densities from grades and the variance of density from the variance of grade.

Measured values in n-dimensional sample spaces may display a significant degree of associative dependence, which is frequently referred to as spatial dependence. Ordered sets of on-stream data at mineral-processing plants almost invariably display a significant degree of associative dependence.

The variance of the arithmetic mean

The arithmetic mean is a functionally dependent variable of a set of measured values with equal weight factors. It can be formulated as follows

$$\bar{x} = (1/n)*x_1 + \dots + (1/n)*x_i + \dots + (1/n)*x_n \quad (1)$$

where
$\bar{x}$ = arithmetic mean of a set of n measurements,
x_i = ith measured value and
n = number of measurements in the set.

The variance of the arithmetic mean can be derived from the formula for the variance of a general function. Intermediate steps in the derivation are based on the premises that the squared partial derivative for each term in Eq. (1) is $(1/n)^2$ and that the variance of the arithmetic mean is the sum of n terms as follows

$$var(\bar{x}) = (1/n)^2*var(x_1) + \dots + (1/n)^2*var(x_i) + \dots + (1/n)^2*var(x_n) \quad (2)$$

for the ordered set of measured values by $F0.95;8;\infty$ = 1.94 gives a lower limit of 496/1.94 = 256 ppm^2 at 95% probability, and dividing it by $F0.99;8;\infty$ = 2.51 gives a lower limit of 496/2.51 = 198 ppm^2 at 99% probability. Hence, the probability is much less than 1% that geostatistical variances of 138.6 and 163.8 ppm^2 are compatible with the variance of 496 ppm^2 for the ordered set.

Because the cost for data acquisition in mineral exploration is high and the geostatistical variances are low, it is understandable that geostatistics is more popular than applied statistics. How the geostatistical theorist contrived to reduce geostatistical variances below confidence limits that probability theory and applied statistics impose merits close examination and scrutiny. After all, systematically low variances would underestimate the risk associated with the measurement of grades and contents of ore deposits.

David (1977) describes how to krige 16 distance weighted averages of the same set of nine measured values, and submits that "writing all the necessary covariances for that system of equations might be a good test to find out whether one really understands geostatistics." A powerful test to find out whether one understands statistics is to calculate the degrees of freedom for David's system of equations. In applied statistics, a set of nine randomly distributed measured values has n – 1 = 8 degrees of freedom, and the ordered set has 2(n – 1) = 16 degrees of freedom. In geostatistics, however, degrees of freedom are deemed irrelevant.

It is tempting to dismiss degrees of freedom as restrictive, even redundant, if it were not for the fact that the requirement of functional independence in probability theory translates into the concept of degrees of freedom in applied statistics. Because kriging variances and covariances of sets of kriged estimates are mathematical aberrations in applied statistics, the question is why kriging became the quintessence of geostatistics.

Rendu (1978) reports that "statistical theory proves that the geostatistical methods of ore reserves estimation are greatly superior to any other method." Given that Clark's geostatistical variances of 138.6 and 163.8 ppm^2 are significantly lower than the variance of 496 ppm^2 for the ordered set, Rendu's praise appears legitimate.

Yet, without the need for functional independence and without the concept of degrees of freedom, geostatistics enjoys an unfair advantage. So much so that Armstrong and Champigny (1989), who noted that kriging variances of sets of kriged estimates become smaller as kriged blocks do, caution that oversmoothed estimates should not be used for calculating recoverable reserves. The authors did not reveal how much smoothing constitutes oversmoothing nor did they report that kriged estimates approach the arithmetic mean of a set of measures values as the distance between the selected position and the measured set increases.

It is beyond dispute that the weighted average is a functionally dependent variable of a set of measured values with different weight factors. It is also beyond dispute that the macrodiamond counts and test masses in Table 1 define one, and only one, mass weighted average but that the uranium values and positions in Table 2 define an infinite set of distance weighted averages.

Computing large sets of kriged estimates (distance weighted averages) of small sets of measure values is tantamount to perpetual motion in data acquisition. Selecting the perfectly smoothed subset of kriged estimates from the infinite set is already a daunting task, but calculating the covariance or the variance of that subset is an exercise in infinite futility indeed.

Summary

The variance is the essence of probability theory and applied statistics. The properties of variances are derived from the variance of the general function as defined in probability theory. The formulas for the variances of the arithmetic mean and the weighted average are derived from the formula for the variance of a general function. Arithmetic means and weighted averages are functionally dependent variables of sets of measured values, but arithmetic means are based on measured values with identical weight factors and weighted averages on measured values with different weight factors.

The distance weighted average and its variance are of crucial importance in mineral exploration. Set theory teaches that two or more measured values of a variable in an n-dimensional sample space define an infinite set of positions and, thus, an infinite set of distance weighted averages. In applied statistics, every distance weighted average of the infinite set has its own variance.

In geostatistics, the distance weighted average metamorphosed into the kriged estimate (Clark, 1979; David, 1977), a simple genesis that evolved into a bizarre variant of Byzantine complexity, a synthesis of probability without functional independence, statistics without degrees of freedom and geomathematics with abandon. This geostatistical variant of probability theory and applied statistics is implemented with dogmatic determination to emulate static stochastic systems of immense intricacy and variability.

David (1977) alludes to the variance of the arithmetic mean as "the famous central limit theorem" and mentions that it gives the variance of the mean of "a group of n independent samples" but omits the variance of the weighted average, which is, in fact, a homologue of the central limit theorem. He predicts correctly that "statisticians will find many unqualified statements" in the "few pages of theory" in which he "firmly grounds the model" and that "the terminology may seem barbarous to statisticians" but appears unaware that the methodology, too, is barbarous.

David refers to "Student's t" but does not mention that tabulated values of the t-distribution are a function of degrees of freedom. He refers to "independent samples" but does not differentiate between spatial dependence and functional dependence. Not surprisingly, his chapter on kriging does not refer to the requirement of functional or mathematical independence. On the contrary, David intimates that the ability to calculate the variance and covariance of a set of functionally dependent variables (kriged estimates) is a test for geostatistical perspicacity.

Armstrong and Champigny (1989) caution against oversmoothing, which seems to imply that the requirement of functional independence can be violated a little, but they fail to define the perfect degree of smoothing. The authors observe that "the kriging variance rises up to a maximum and then drops off," and submit that this extraordinary behavior is described by Brooker (1992) who praises its robustness.

Rendu (1994) worries about "an endless list of

TABLE 3

Positions and values for hypothetical uranium data - Part two.

Datum number	Easting, m	Northing, m	Grade, ppm	Distance, m	Weight factor	Δa ppm
5	1,244	713	320			
4	1,265	704	280	23.2	1.630	40
1	1,271	711	400	9.1	0.636	120
2	1,280	713	380	9.4	0.664	20
3	1,268	722	450	15.2	1.070	70

Source: *Practical Geostatistics* (Clark, 1979).

Substituting in Eq. (5) the uranium values and the weight factors of Table 2 gives a distance weighted average of 0.319*400 + 0.137*380 + 0.217*450 + 0.229*280 + 0.098*320 = 373 ppm. Substituting in Eq (6) the weight factors of Table 2 gives the following terms for the variance of the distance weighted average of 373 ppm

$$var(\bar{x}) = 0.319^2 var(x_1) + 0.137^2 var(x_2) + 0.217^2 var(x_3) + 0.229^2 var(x_4) + 0.098^2 \, var(x_5) \qquad (7)$$

Because the variance for each term is unknown, either the variance of the randomly distributed set or the variance of the ordered set should be used in all terms. Substituting the variance of 4,480 ppm^2 for the randomly distributed set of uranium values gives

$$var(\bar{x}) = (0.319^2 + 0.137^2 + 0.217^2 + 0.229^2 + 0.098^2)*4{,}480 = 1{,}029 \text{ ppm}^2$$

This variance is equivalent to a standard deviation of √1,029 = 32.1 ppm, a 95% confidence interval of t0.95;4*32.1 = 2.776*32.1 = ±89 ppm or 89*100/373 = ±24% and a 95% confidence range with a lower limit of 373 – 89 = 284 ppm and an upper limit of 373 + 89 = 462 ppm.

The position of the distance weighted average of 373 ppm is within the sample space defined by this set of uranium values. Therefore, it makes sense to check whether the degree of associative dependence between ordered values is statistically significant and, thus, whether the probability to encounter continued mineralization within this sample space is high. The question of whether the ordered set exhibits spatial dependence within its two-dimensional sample space can be assessed by applying analysis of variance (ANOVA) to the variances of randomly distributed and ordered sets of measured values.

Table 3 lists the same data sets as Table 2 but ordered in such a manner that a systematic walk, which calls on each of the five positions only once, covers the shortest possible distance.

The variance of 2,160 ppm^2 is a function of the differences between ordered measurements and the distances between positions. Substituting in Eq. (6) the variance of 2,160 ppm^2 and the weight factors of Table 2 gives

$$var(\bar{x}) = (0.319^2 + 0.137^2 + 0.217^2 + 0.229^2 + 0.098^2)*2{,}160 = 496 \text{ ppm}^2$$

The variance of 496 ppm^2 for the distance weighted average of 373 ppm is equivalent to a standard deviation of √496 = 22.3 ppm, a 95% confidence interval of t0.95;8*22.3 = 2.306*22.3*2.306 = ±51 ppm or 51*100/373 = ±14% and a 95% confidence range with a lower limit of 373 – 51 = 322 ppm and an upper limit of 373 + 51 = 424 ppm. The value of t0.95;8 = 2.306 is listed in the t-distribution at p = 0.95 and df = 8 and reflects that an ordered set of five measured values has 2(n – 1) = 8 degrees of freedom.

The question of whether variances of 4,480 ppm^2 for the randomly distributed set and 2,160 ppm^2 for the ordered set are statistically identical is solved by comparing their ratio of 4,480/2,160 = 2.07 with tabulated F-values. The F-ratio of 2.07 exceeds neither F0.99;4;8 = 7.01 (tabulated in the F-distribution for p = 0.99, dfr = 4 and dfo = 8) nor F0.95;4;8 = 3.84 (tabulated at p = 0.95 with the same degrees of freedom). Therefore, the degree of spatial dependence in this two-dimensional sample space is not statistically significant.

Even though these variances do not differ significantly, the lowest may be used to calculate precision estimates, but it would be erroneous to claim that spatial dependence results in a significantly higher degree of precision for the distance weighted average of 373 ppm. The expression "marginally more precise" reflects in an appropriate manner that the lowest variance gives a numerically more desirable degree of precision than the highest variance does.

Mathematical analysis should not be applied to the difference between statistically identical variances, because the difference of 4,480 – 2,160 = 2,320 ppm between the variances of the randomly distributed and ordered sets is only a random number. In geostatistics, such differences are entered into "*smoothing relationships*" to predict tonnages and grades (David, 1989).

Clark (1979) refers to inverse distance and extension estimates and reports the precision of the distance weighted average in terms of 95% confidence intervals, and lower and upper limits of 95% confidence ranges. Table 4 lists confidence limits calculated from statistical and geostatistical variances.

Because geostatistics is unencumbered with degrees of freedom, it would be imprudent to apply ANOVA to the geostatistical variances in Table 4, but it can be applied to obtain asymmetric lower limits of confidence ranges for variances. Dividing the variance of 496 ppm^2

TABLE 4

Precision estimates for hypothetical uranium data.

Methodology	Variance	95% CI	95% CRL	95% CRU
Applied Statistics:				
Random	1,029	±89	284	462
Order	496	±51	322	424
Geostatistics:				
Inverse Distance	138.6	±23.6	349	396
Extension	163.8	±25.6	340	392

Source: *Practical Geostatistics* (Clark 1979).

When a set of measured values of a variable is obtained by applying the same measurement procedure to the same sample space, the variance of the set is an unbiased estimate for the variance of each term, and the formula for the variance of the arithmetic mean becomes

$$var(\bar{x}) = n*[(1/n)^2*var(x)] \quad (3)$$

which implies that

$$var(\bar{x}) = (1/n)*var(x) = var(x)/n \quad (4)$$

Equation (4) is the central limit theorem. It can be used to show that the central tendency of samples selected of multinomial, binomial or Poisson distributions tends toward the normal distribution. For example, if n values are measured in randomly selected samples of different sample spaces, the variance of each term in Eq. (2) should be weighted with the factor wi = ni/n, the ratio between n_i, the number of measured values in *ith* sample space and n, the number of measured values in all sample spaces.

The variance of the weighted average

If wi, the weight factor for each term, is defined such that $\Sigma wi = 1$ for the function, then the sum of all weight factors 1/n for the arithmetic mean meets this prerequisite. The weighted average is identical to the arithmetic mean if each wi is equal to 1/n. Substituting the weight factors in Eq. (1) gives the following formula for the weighted average:

$$\bar{x} = w_1*x_1 + ... + w_i*x_i + ... + w_n*x_n \quad (5)$$

where
$\bar{x}$ = weighted average of a set of n measurements,
w_i = ith weight factor,
x_i = ith measured value and
n = number of measurements in the set.

Because the squared partial derivative for the ith term is wi^2, the variance of the weighted average of a set of n measured values with weight factors wi is

$$var(\bar{x}) = w_1^2*var(x_1) + ... + w_i^2*var(x_i) + ... + w_n^2*var(x_n) \quad (6)$$

Numerical examples show how to apply the formula for the variance of the weighted average in sampling practice. One deals with a mass weighted average of macrodiamond counts in test samples of different mass, and the other deals with a distance weighted average of uranium values in geochemical samples selected at different positions in a two-dimensional sample space.

In practical applications, the set of diamonds detected in a test sample is often grouped into subsets of macro- and microdiamond counts, but they may be grouped into more than two classes. In this example, only macrodiamonds counts are used to calculate the mass weighted average and its variance.

TABLE 1

Different macrodiamond counts and sample masses.

Hole number	Macro count	Mass, Kg	Weight factor
1	4	16	0.098
2	2	12	0.073
3	15	64	0.390
4	19	72	0.439
SUM	40	164	1.000

Table 1 lists a set of macrodiamond counts determined in test samples of different mass.

The macrodiamond counts and sample masses in Table 1 have one, and only one, mass weighted average, namely, 0.098*4 + 0.073*2 + 0.390*15 + 0.439*19 = 14.7 or approximately 15 macrodiamonds. The variance of this mass weighted average is $0.098^2*4 + 0.073^2*2 + 0.390^2*15 + 0.439^2*19 = 6.0$, for a standard deviation of $\sqrt{6.0} = 2.45$, a 95% confidence interval of 2*2.45 = ±4.9 or 4.9*100/14.7 ± 33% and a 95% confidence range with a lower limit of 14.7 – 4.9 ≅ 10 and an upper limit of 14.7 + 4.9 ≅ 20 macrodiamonds.

The previous example shows how to calculate the variance of the mass weighted average of a set of macrodiamonds counts determined in test samples of different mass. The next example is based on a hypothetical set of uranium data (Clark, 1979) and shows how to calculate the variance of a distance weighted average of a set of values measured at different positions within a two-dimensional sample space.

Table 2 lists the measured values and their positions, the position for which the weighted average is calculated, the distances between the measured values, the inverse distance for each position and its weight factor.

The arithmetic mean of the set of uranium values in Table 2 is 366 ppm, and the variance of the randomly distributed set is 4,480 ppm². The central limit theorem teaches that the variance of the arithmetic mean is 4,480/5 = 896 ppm², for a standard deviation of $\sqrt{896} = 29.9$ ppm, a 95% confidence interval of $t0.95;4*29.9 = 2.776*29.9 = \pm 83$ ppm or 83*100/366 = ±23% and a 95% confidence range with a lower limit of 366 – 83 = 283 ppm and an upper limit of 366 + 83 = 449 ppm. The tabulated t-value of 2.776 at 95% probability reflects that a set of five randomly distributed measurements has 5 – 1 = 4 degrees of freedom.

TABLE 2

Positions and values for hypothetical uranium data - part one.

Datum number	Easting, m	Northing, m	Value, ppm	Distance m	Inverse distance	Weight factor
1	1,271	711	400	6.6	0.1523	0.319
2	1,280	713	380	15.2	0.0656	0.137
3	1,268	722	450	9.6	0.1038	0.217
4	1,265	704	280	9.1	0.1094	0.229
5	1,244	713	320	21.3	0.0469	0.098
Selected Position	1,265	713				
Weighted Average			373			

Source: *Practical Geostatistics* (Clark, 1979).

kriging methods" but wonders why skepticism remains. Perhaps the skeptics wonder which kriging method is the least biased, why kriging variances give the most precise estimates, or whether kriging variances of small sets of kriged estimates violate the requirement of mathematical independence as much as those of large sets do. Nonetheless, Rendu pronounces that "geostatistics has come of age," and predicts that "it is here to stay with all its strengths and weaknesses."

In the section on sampling variograms, Gy (1979) points out that the denominators in the formulas for the variances of randomly distributed and ordered sets of measured values are degrees of freedom. Even though Gy's works are invariably quoted whenever geostatisticians require a reference to sampling theory or practice, the significance of degrees of freedom continues to elude the geostatistical theorist. In fact, a geostatistical reviewer recently postulated that "degrees of freedom is an older terminology that is not relevant to the modern development of statistics".

Matheron (1963) hails geostatistics as "a new science" but Philip and Watson (1986) dismiss it as "a sham." Armstrong (1992) editorializes that Philip and Watson's paper is "high on journalistic style and low on scientific content" and is not surprised that "neither set of authors succeeded in getting their ideas published in the usual way in a technical journal." Armstrong commiserates with authors who "disagree violently with geostatistics" and "express opinions that run against the popular view" and whose papers are rejected because they "do not back up their opinions with scientific fact".

The scientific facts are that kriging variances and covariances of sets of kriged estimates violate the fundamental requirement of functional independence of probability theory and that the concept of degrees of freedm is and will always be relevant in applied statistics.

Other scientific facts are that geostatistical "variances" are mathematical aberrations that cannot possibly give unbiased precision estimates as a measure for the risk associated with the measurement of grades and contents of ore deposits. Applied statistics, by contrast, gives unbiased precision estimates in terms of confidence limits for grades and contents as intuitive and reliable measures for the risk that the progression from prospect to producing mine entails.

The geostatistical theorist ought to explain how it is possible to design a geostatistical model on the basis of widely spaced exploration drill-hole grades when closely spaced blasthole grades do not even display a significant degree of spatial dependence. ■

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